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Original Paper

Pre-service Teachers’ Reasoning and Sense Making of Variables

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Abstract

Researchers note that algebraic reasoning and sense making is essential for building conceptual knowledge in school mathematics. Consequently, pre-service teachers’ own reasoning and sense making are useful in fostering and developing students’ algebraic reasoning and sense making. This article explores the forms of reasoning and sense making that pre-service mathematics teachers exhibit and use in the process of analysing problem- posing tasks with a focus on first-degree equations. The research question concerns the characteristics of the problem-posing tasks used for reasoning and sense making of first- degree equations as well as the characteristics of pre-service teachers’ reasoning and sense making in problem-posing tasks. The analyses are grounded in a post-structuralist philosophical perspective and variation theory. Sixty-six pre-service primary teachers participated in the study. The results show that the characteristics of reasoning in problem- posing tasks and of pre-service teachers are selecting, exploring, reconfiguring, encoding, abstracting and connecting. The characteristics of sense making in problem-posing tasks and of pre-service teachers are recognition, relationships, profiling, comparing, laddering and verifying. Beside this, the connection between reasoning and sense making is rich in line of flight in problem-posing tasks, while the connection is rich in line of rupture for pre-service teachers.

Keywords: first-degree equation, problem posing, reasoning, rhizomatic assemblage, sense making, variation theory

1. Introduction

Research demonstrates that algebra has a key role in understanding mathematical concepts and has made significant contributions to the development of technology. Despite the importance given to algebra, failure rates in algebra continue to be high, especially the transition to formal algebraic reasoning and in translating the verbal expressions into algebraic expressions (Cañadas & Castro, 2007; Daud & Ayub, 2019; Kaput, 1995; Küchemann, 1978). Researchers note that algebraic reasoning is essential for building conceptual knowledge in school mathematics, and they often express the teachers’ impact on creating change and developing the reasoning students need for understanding, thereby leading to success in algebra (e.g., Arcavi, Drijvers & Stacey, 2017; Ayalon, Watson, & Lerman, 2015; Hoch & Dreyfus, 2005, 2010; Kieran, 2006). Consequently, the pre-service teachers’ own reasoning and sense making of compulsory algebra (Swedish compulsory education comprises nine years of schooling, for children ages 7 to 16) are useful in fostering and developing students’ algebraic reasoning and sense making. To avoid confusion, the term ‘teacher’ is used to denote a person actively engaged in teaching in compulsory school; ‘pre-service teachers’ represent university students in teacher programmes and ‘students’ are those studying in compulsory school.

The use of problem posing is listed in the literature as an opportunity to overcome students’ mistakes. *Problem posing* is the process of formulating and expressing a problem within the domain of mathematics (Cai & Hwang, 2020; Cai & Hwang, 2002; English, 2003; Silver, 1994; Tichá & Hošpesová, 2009). Pre-service teachers becoming mathematics teachers also need to know how to use problem posing and to understand how this process influences students’ learning. In the literature, problem posing is used to develop teachers’ competencies, and as a tool for sense making (Cifarelli & Sevim, 2015; Xu, Cai, Liu & Hwang, 2020). However, there are a few studies that focus on pre-service teachers and the relationship between reasoning and sense making in connection with first-degree equations with one unknown (e.g., Işık & Kar, 2012b). In this context, this article focuses on the analysis of problem posing in relation to first-degree equations with one unknown.

This article explores the features of pre-service teachers' reasoning and sense making in algebra, specifically using first-degree equations. The main purpose of the research is to investigate the forms of reasoning and sense making that pre-service mathematics teachers exhibit and use in the process of analysing problem posing, with a focus on first-degree equations. The following research questions served as a guide in the analysis of data: What are the characteristics of the problem-posing tasks used for reasoning and sense making of first-degree equations? What are the characteristics of pre-service teachers' reasoning and sense making in problem-posing tasks?

2. Literature Review

Algebraic expressions and equations

Over the past few decades research has been dedicated to providing a view of what school algebra is and which aspects belong to this content. They found that algebra is: (1) generalized arithmetic (e.g., Bell, 1996; Kaput, 1995; Usiskin, 1999); (2) symbolic manipulation (e.g., Kaput, 1995; Olteanu, 2007); (3) forming and solving equations (e.g., Bell, 1996; Olteanu, 2015; Usiskin, 1999); (4) relationships between quantities (e.g., Cooney, Beckmann, & Lloyd, 2010; Kaput, 1995; Olteanu, 2015; Usiskin, 1999); (5) the study of structure (e.g., Kaput, 1995; Olteanu, 2015, 2016; Usiskin, 1999); and (6) an activity (Kieran, 1996; Lee, 1997; Olteanu 2016). There is also a great deal of research focusing on students' algebraic thinking, students' understanding of algebraic concepts and their common errors and misconceptions, as well as teaching strategies to address these errors and misconceptions (e.g., Leinhardt, Zaslavsky, & Stein, 1990; MacGregor & Stacey, 1997; Matz, 1982; Olteanu, 2007).

In mathematics, algebraic expressions and equations are encountered very often, and equations and the finding of solutions form the base of algebra. However, these two mathematical terms are not the same. One big difference between an algebraic expression and equation lies in their arrangement, namely in the explanation of what they represent. Research related to algebraic expressions and equations includes studies that explore the concept of variables and their meanings (e.g., Küchemann, 1978; MacGregor & Stacey, 1997; Usiskin, 1999; Olteanu, 2007), equivalence and the equal sign (e.g., Asquith, Stephens, Knuth, & Alibali, 2007; Knuth, Stephens, McNeil, & Alibali, 2006), and extrapolation techniques, such as linearity and generalization (e.g., Matz, 1982; Olteanu, 2007). Research related to the concept of variables and variable meaning is of particular interest in this study because it can inform the types of knowledge that pre-service teachers need to teach algebra.

Variables can represent generalized numbers, unknowns, arguments, parameters, or arbitrary symbols (Küchemann, 1978; Olteanu 2007; Usiskin, 1999). Common student misconceptions related to variables include viewing variables as abbreviations or labels (e.g., Küchemann, 1978), viewing all variables as specific values (e.g., Asquith et al., 2007), and being unable to accept expressions containing variables as final answers to problems (e.g., Collis, 1975). As a consequence of these misconceptions, many students find it hard to understand the idea of equivalence or the role of the equal sign in mathematical sentences (e.g., Asquith et al., 2007; Baroody & Ginsburg, 1983; Powell 2015). The difficulties experienced in translating verbal expressions into algebraic expressions were also emphasized in studies about algebra (e.g., Herscovics & Kieran, 1980; Rosnick, 1981).

To overcome students' misconceptions, research suggests approaches that, among other things, focus on building on students' prior knowledge (e.g., Arcavi et al., 2017), using multiple representations (e.g., Goldin & Shteingold, 2001; Olteanu, 2007), choosing a wide range of rich examples, tasks, and questions (e.g., Leinhardt et al., 1990; Olteanu, 2015) or using patterns of variation to give the students the opportunity to discern the critical aspects of the object of learning (Olteanu, 2016). Olteanu (2016) considered an *object of learning* to be part of an event that is formed through teacher–student communication. The object of learning is the content that the teacher intends to teach, as well as how the students are expected to make sense of and use the content (Marton & Booth, 1997).

Problem posing

Research on problem posing is not new, but there is new research that examines this subject carefully (e.g., Cai & Hwang, 2020). Researchers have found that there is a correlation between students' success in problem solving and their problem-posing abilities, and that teachers could use problem posing to better understand their students' mathematical thinking (Cai et al., 2013).

There are some studies concerning teachers and pre-service teachers posing and solving problems corresponding to division operations with a remainder (Chen, Dooren, Chen & Verschaffel, 2011), fractional division operation (Işık & Kar, 2012a), symbolic expressions of fraction multiplication (Luo, 2009), and to equations (Işık & Kar, 2012b). Results from these studies show that pre-service teachers had limited understanding about the content of a good mathematical problem (Chen et al., 2011); they were unable to construct appropriate word problems for the given symbolic expressions of fraction multiplication (Luo, 2009); they used incorrect translations of mathematical notations into problem statements, gave unrealistic values to the unknowns, and changed the equation structure (Işık & Kar, 2012b). The way in which pre-service teachers pose story problems based on equations was studied by Stephens (2003) and Walkington (2010). The results show that problem-posing was a difficult task for students because it required them to make meaning from symbols in a way that simply solving story problems does not; that pre-service teachers struggled to reason structurally about the equation, and that they struggled to understand the role of variables as unknown quantities. However, such studies report that both pre-service and in-service teachers can pose interesting and important mathematical problems (e.g., Stickles, 2011).

Cai and Hwang (2020) specify that a problem-posing tasks require teachers or students to generate new problems and questions based either on given situations or on mathematical expressions or diagrams. They suggest that further research could vary the kinds of tasks to probe how different conditions might produce different ideas and perspectives on what counts as a good task. As far as the authors are aware, there are few studies that discuss the pre- service teachers' reasoning and sense making when they analyse proposed problem-posing tasks. Therefore, in this study, the authors analyse reasoning and sense making of pre-service teachers in solving proposed problem-posing tasks.

Reasoning and sense making

'Reasoning' is used in the literature in different ways and there is no clear definition (e.g., Ball & Bass, 2003; Duval, 2002; Lithner, 2008; Martin et al., 2009; Olteanu, 2020). For example, Ball and Bass (2003, 28) state that 'mathematical reasoning is no less than a basic skill'; Duval (2002) considers only strict proof as mathematical reasoning, and Olteanu (2020) considers reasoning as a way of selecting, exploring, reconfiguring, filtering, encoding, abstracting and connecting to highlight associations and relationships between different content. Further, the term 'sense making' is used with different meanings. For example, Van Velzen (2016) used sense making to refer to deepening one's understanding by applying mathematical concepts. Weick et al. (2005) used sense making as the individuals' decisions and actions based on their understanding, while Klein et al. (2006) and Olteanu (2020) argued that sense making is a continuous effort to understand connections to anticipate their trajectories and act effectively. According to Olteanu (2020), these connections are generated by: *recognition of similar attributes between different objects of learning* (e.g., the use of variables in algebraic expressions); *relationships between concepts* (e.g., algebraic expressions and equations); *profiling* (e.g., features that influence the selection of aspects of content); *comparing variables*; *explaining the link between concepts*; *laddering one piece of information to the next one*; and *verifying*.

Researchers have demonstrated that sense making is supported in the classroom environment by interaction, negotiation and discussion, which create meaningfulness and understanding (Boaler, 2014; Schoenfeld, 2017); using student thinking through involving verbal interactions (Leatham et al., 2015); and using tasks that give students the opportunity to extend their ideas, explain and justify mathematical concepts (Mueller, Yankelewitz & Maher, 2014; Olteanu, 2020). Olteanu (2020) used concepts from variation theory (e.g., Marton, 2015) in combination with a post-structuralist philosophical perspective (Deleuze & Guattari, 1987) to analyse what features of tasks allow students to experience reasoning and sense making and what features are discerned in students' reasoning and sense making. A strong connection between reasoning and sense making requires particular conditions: (a) thoughtful teacher interventions if lines of rupture are created when experiencing intended critical aspects; (b) creating rhizomatic tasks; (c) identifying critical aspects; and (d) using patterns of variation (Olteanu, 2020). In this context *lines of rupture* are used to explain the limitations imposed by the students' experience, revealing new ways of thinking and acting (Olteanu, 2020). 'Critical' in critical aspects refers to a critical difference in the learners' ways of grasping and becoming acquainted with the object of learning (Olteanu & Olteanu, 2012). The critical aspects consist of three components that interpenetrate: the intended (the aspects of the content that teachers intend to present in the classroom), enacted (the aspects of the content that teachers actually focus on during the lesson), and lived aspects (the aspects of the object of learning that the students distinguish during or after a lesson) (Olteanu, 2015, 2016).

In this article, the object of learning is the pre-service teachers' reasoning and sense making when they work with problem posing in relation to first-degree equations with one unknown. The focus is only on the intended and lived critical aspects because they can inform about the characteristics of the problem-posing task as well as the pre-service teachers' reasoning and sense making.

Theoretical framework

In this article, the unit of analysis has three dimensions: reasoning, sense making and critical aspects. According to Olteanu (2020), reasoning and sense making are closely related to each other and to these dimensions in the manner of a 'rhizome' (Deleuze & Guattari, 1987). The rhizome concept is used to perceive the critical aspects as *assemblages* of words, ideas, concepts, and countless other things that are related to one another but that are also distinct (Olteanu, 2020). According to Freitas (2012), each assemblage is a constellation of heterogeneous elements and is understood as comprising processes or connections. According to Deleuze and Guattari (1987), the characteristics of a rhizome, among other things, are: (1) *connection* (new connections are created at every point); (2) *heterogeneity* (associating and linking elements of quite different types); (3) *multiplicity* (creating a whole with specific properties that cannot be explained by adding the separate properties of individual parts); (4) *rupture* (the ability of growth to start again along an old line or along a new line if an old line is broken at any place); and (5) *cartography* (no beginning or end, but all points may serve as points of departure).

The theoretical model presented by Olteanu (2020) enables the study of sense making and reasoning as rhizomatic assemblages. Three lines contribute to growing the assemblage. The lines of segmentarity (lines that outline classification), the lines of flight (the parts of the assemblage that escape the structure of which they are part, serving to connect such an assemblage to that which is outside itself), and the lines of rupture. The *lines of rupture* are 'as if something carried us away, across our segments, but also across our thresholds, towards a destination which is unknown, not foreseeable, not pre-existent' (Deleuze & Parnet, 2002, 125). The lines of flight are associated with the new, with change and reconstruction (Freitas, 2012). As previously mentioned, *sense making* is a continuous effort to understand connections in order to anticipate their trajectories and act effectively, and *reasoning* is a way of selecting, exploring, reconfiguring, filtering, encoding, abstracting, and connecting (Olteanu, 2020). *Selecting* is the process of searching for similarities and differences in order to choose tools (concrete, pictorial, and symbolic) by making careful decisions – for example, selecting a variable as an unknown number. *Exploring* is the process of searching for similarities and differences in order to find out more information about an equation. *Reconfiguring* is also the process of searching for similarities and differences in order to change the structure or arrangement of an equation. *Filtering* is the process for choosing or removing a particular type of information and *encoding* refers to the use of a word, a phrase or mathematical symbol in the correct way. *Abstracting* is the process that involves the identification of common and important features that result in a new concept – for instance, the identification of common and important features for an algebraic expression and an equation. *Connecting* is the process of highlighting associations and relationships between different content, concepts, or ideas – for instance, the connections between the two problems posed.

The moment of rupture corresponds to the way of understanding a phenomenon, and this in turn is directly connected to which critical aspects are discerned (Olteanu, 2020). Further, the qualitative differences between ways of experiencing give the opportunity to identify the lines of flight, segmentarity and rupture that organize the critical aspects through constant decomposition and reconstruction. Variation theory claims that to develop a particular capability (e.g., reasoning), one must focus simultaneously on all critical aspects of the object of learning. This perspective also claims that one can only focus on what is discerned, discern what is experienced as varying, and experience variation if one has previously experienced instances of variation (e.g., Marton, 2015). To understand the connectivity between sense making, reasoning, and critical aspects the focus in this article is on capturing different types of lines and the conditions of their specific exposures. The connection between critical aspects, reasoning and sense making is visualized in Figure 1. In this figure, the lines of segmentarity are represented as a long line and a dot, the lines of flight as a solid line, and the lines of rupture as dots.

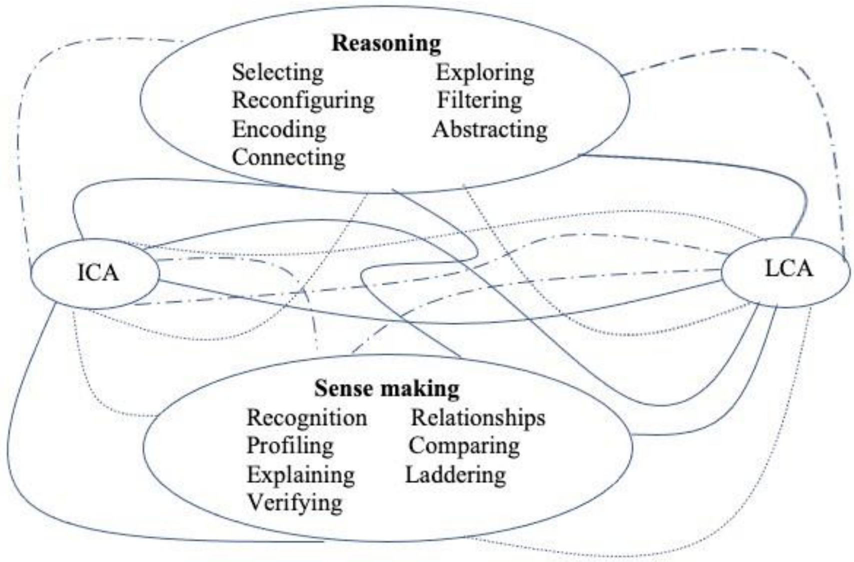


Figure 1. Critical Aspect, Reasoning, Sense Making and the Lines of Connections

From this philosophical perspective, the difference is a singularity (turning points and points of inflection in a topological way) at the level of ideas, and repetition is always affected by an order of difference (Olteanu & Olteanu, 2018). Deleuze (1994) specified that the concepts of repetition and difference will intersect, 'one concerning the essence of repetition, the other the idea of difference' (p. 31). Repetition is never the reproduction of the same, but the repetition of the different (Deleuze, 1994).

3. Methodology

Context

This research has been developed with pre-service primary teachers in a mathematics teacher education course. This course is part of the first year of a four-year primary teacher education program at a big public university in a Nordic country. At present, the mathematics teacher education course comprises arithmetic, algebra, geometry, probability, and statistics. There are three parts to each of the four sections: a theoretical part, a practical part, and experience in school. In the theoretical part, the focus is on providing pre-service teachers with conceptual tools and theoretical constructs that have been generated from research in mathematics teacher education. The practical part is dedicated to putting these tools into practice using activities proposed by teacher educators, and the experience part is the pre- service teachers' participation in authentic school activities.

Data collection

In this study, the data was collected when pre-service elementary teachers were solving mathematical tasks posed in algebra, but just one task will be presented here. There were two reasons for the choice of this task: first, it encompassed the use of problem posing; second, the concept in this task (first-degree equations) is commonly used in the compulsory algebra classroom. The data for this article consist of pre-service teachers' written solutions. Some studies have pointed out pre-service teachers' difficulties with problems. For this reason, we decided to use a task in which the first-degree equation as well as the problem posing are explicitly formulated.

Participants

This study is part of ongoing research carried out with pre-service teachers enrolled in the abovementioned mathematics teacher education course. All the pre-service teachers were older than 19, from diverse socio-economic backgrounds and attended class for all 10 weeks of the semester, including the seven weeks in which data was collected. Sixty-six pre-service primary teachers participated in an anonymous written exam and were informed about our research (characteristics, aim, confidentiality issues, etc.). In this study the students' tasks from an anonymous written exam were analysed. The students are not involved direct in the study.

Data analysis procedure

The data analysis adopted a qualitative/interpretative approach and was carried out in seven steps. In the first step, the tasks were analysed with respect to the intended critical aspects, the dimension of variation, and the opportunity for reasoning and sense making. In the second step, the lines of flight, segmentarity and rupture were identified. In the third step, the pre- service teachers' written solutions were analysed by two researchers independently and simultaneously. The written solutions were analysed using the categorical analysis technique, which is a type of content analysis. In this process, each researcher formed categories according to the problems posed. In the fourth step, their analyses were compared and a consistency of 92% was achieved in the classification of the difficulty types identified. In the fifth step, the common identifications were selected, and the problematic identifications were discussed in order to be selected or rejected. In the sixth step, the identified categories were triangulated with sense making, reasoning, and the lines of connection (the lines of segmentarity, the lines of flight, and the lines of rupture (Figure 1).

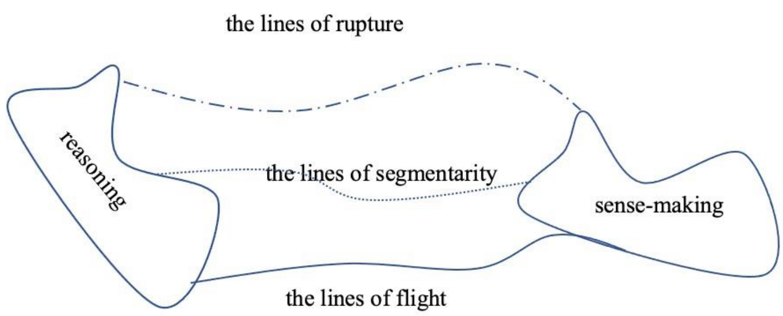


Figure 2. Reasoning, Sense Making and the Lines of Connection

In analysing reasoning and sense making, the typology used by Olteanu (2020) was used. In the last step, the aspects discerned by the pre-service teachers in the two tasks were analysed.

4. Results

The characteristics of the tasks

The problem-posing task chosen in this article is based on the idea of comparing three problem-posing structures. The problem-posing task gives pre-service teachers the opportunity to discover the relationships between similar and different problem-posing structures in relation to a first-degree equation with one unknown on both sides of the equal sign. The problem-posing task is:

> Which problem matches to the equation 200 - (x + 60) = x?
>
> Explain how you thought (in points A, B and C) and provide an argument for your thoughts.

A. Maria has SEK 200. After buying a gift for her brother and receiving

SEK 60 from her father, does Maria have left as much as the gift cost?

B. At a company party there were 200 people before the finance department went home. The production department of 60 people also went home. How many were left at the party?

C. Susanna sells freshly baked bread. After selling some loaves to Bengt and 60 to another customer, Susanna still has as many loaves as she sold to Bengt. How many loaves does she have left if she had 200 loaves at the beginning?

The first intended critical aspect (ICA1) is to discern x as a particular (but unknown) number. This critical aspect gives the opportunity for pre-service teachers to use two components of reasoning; namely, to explore the equation and recognize that an equation is made up of two expressions connected by an equal sign. Also, this critical aspect gives the opportunity for sense making – reconfiguring the equation, encoding the equation, and laddering one piece of information to the next one. The second intended critical aspect (ICA2) is about the translation of the problem-posing task into an algebraic expression or equation, and the third intended critical aspect (ICA3) is the connection of this translation to the given equation. Those critical aspects give the opportunity for pre-service teachers to reason by selecting a variable that represents an unknown number or a general number, exploring the information

in problem posing, encoding an equation or an algebraic expression, abstracting common and important features that result in an equation or an algebraic expression, reconfiguring the equation structure, and connecting different words that are used to indicate equivalence in problems. The reasoning results in sense making of the use of letters in algebra (recognition), of the relationships between algebraic expression and equation, of the features that influence the selection of problem posing (profiling), of the variables (comparing), of the laddering of one piece of information to the next one, and verifies the result with the original equation. The rhizomatic assemblage in the purposed task is constituted of the lines of flight (Figure 3).

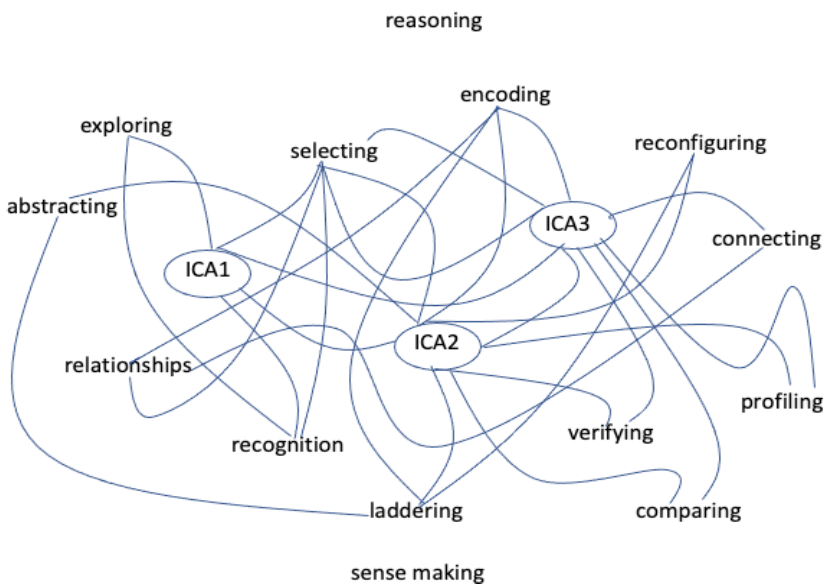


Figure 3. Intended Critical Aspects, Reasoning and Sense Making

The task assemblage has, with focus on reasoning and sense making, produced different opportunities for pre-service teachers to identify similarities and differences between first- degree equations and algebraic expressions.

The characteristics of pre-service teachers' reasoning and sense making

The first lived critical aspect (LCA1) is to think of *x* as a particular (but unknown) number; 90% of the pre-service teachers start by solving the equation 200 - (x + 60) = x, and 10% start with exploring the problem-posing task. Of the pre-service teachers that start by solving the equation, 70% solve the equation correctly, and 20% have trouble with the distributive property and the invisible one before the parentheses. The reasoning of those pre-service teachers shows that they do not discern how to encode the equation, how to ladder one piece of information to the next one, and consequently to reconfigure the equation. A line of rupture is created in the task assemblage.

Regardless of how to approach the task, all pre-service teachers explore the problem-posing task, and 40% frame that alternative A corresponds to the given equation, 25% frame that it is alternative B, and 30% frame that it is alternative C. Alternative A is chosen as a result of the pre-service teachers' reasoning about the encoding of the equation. For instance, in the filtering process most pre-service teachers write down an equation that does not correspond to the sentence "After buying a gift for her brother and receiving SEK 60 from her father […]" (Figure 4). This type of reasoning involves cases in which the operations and parentheses in an equation are incorrectly translated in problem statements (explaining, profiling). This way of reasoning shows that pre-service teachers do not discern the translation of the problem- posing task into an equation (LCA2). A line of rupture is created in the task assemblage.

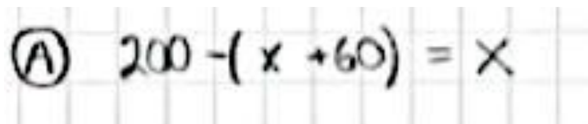
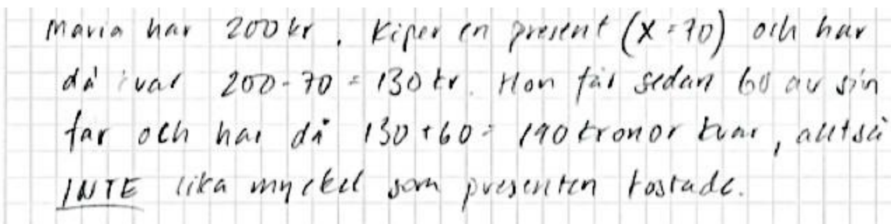


Figure 4. Filtering and Encoding

There are also pre-service teachers that connect the solution for the equation 200 - (x + 60) = x to the problem posing in alternative A (Figure 5) and draw a correct conclusion. In most of these cases, pre-service teachers verify the solution of the given equation in a new context. In this way a line of flight is built up in the task assemblage. Those pre-service teachers explain the link between the algebraic and arithmetic equals sign by laddering one piece of information to the next one.



Translation: Maria has 200 SEK. She buys a gift (x = 70) and has left 200 -70 = 130 SEK. She then receives 60 from her father and has 130 + 90 SEK left, thus not as much as the gift cost.

Figure 5. Connection

There are also pre-service teachers that reconfigure the initial equation and thereafter compare the two equations (Figure 6). A line of flight is built up in the task assemblage.

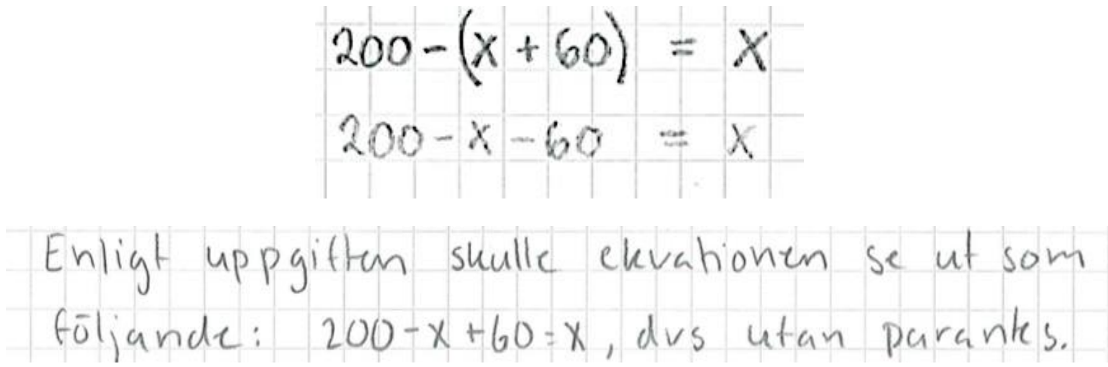
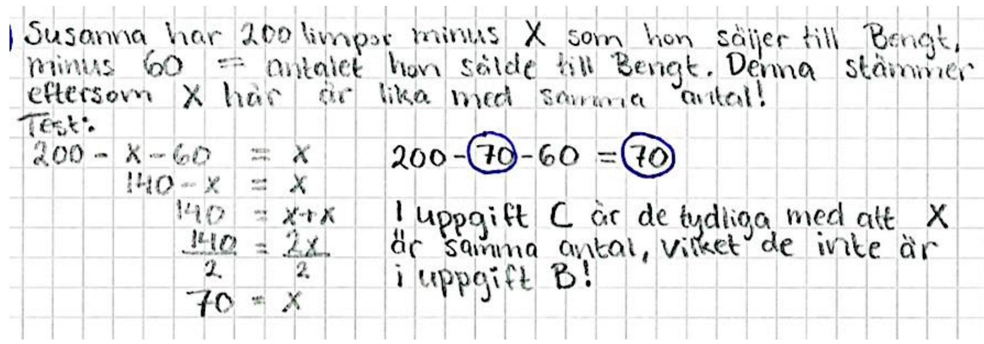


Figure 6. Reconfigure and Compare

All pre-service teachers that chose alternative B failed to translate the posed problem into an algebraic expression (LCA3). They do not discern the difference between an equation and an expression nor the use of a variable as an unknown number or as varying variable. In this way a line of rupture is built up in the task assemblage between reasoning (exploring, filtering, connecting) and sense making (recognition, relationships, comparing, explain, laddering).

The pre-service teachers that chose alternative C arrive at the mathematically correct answer, and there is no mathematical error. The reasoning involves selecting a variable as an unknown number, exploring the similarity between the given equation and the equation corresponding

to the problem posed, encoding the equation, reconfiguring the equation to find the solution, and connecting relationships between different content and ideas. This way of reasoning leads in turn to sense making of the translation of a situation (recognition) into symbols by explaining what the symbol represents and how (laddering) and verifying the solution (Figure 7). The line of flight connects reasoning and sense making in the task assemblage.



Translation: Susanna has 200 freshly baked loaves minus x which she sells to Bengt, minus 60 = the number she sells to Bengt. This is correct because x here is equal to the same number.

Figure 7. LCA2, LCA3, Reasoning and Sense Making

To summarize the characteristics of pre-service teachers' reasoning and sense making, Figure 8 is an illustration of the lines of flight and rupture connected to lived critical aspects in the problem-posing task.

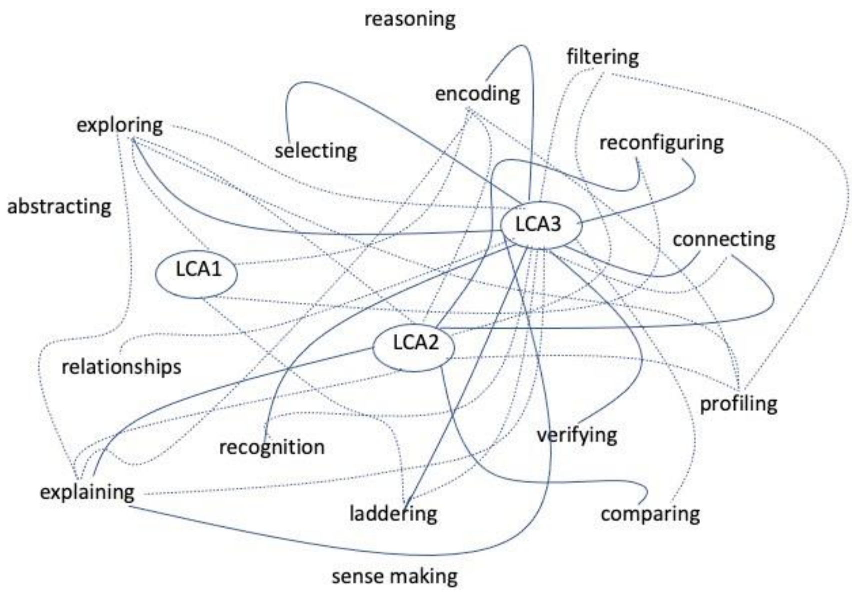


Figure 8. Lived Critical Aspects, Reasoning and Sense Making

The rhizomatic assemblage is characteristic of several lines of rupture. This indicates that problem posing can be used to better understand strengths and weaknesses in pre-service teachers' reasoning and sense making.

5. Discussion

This article explores the features of pre-service teachers' reasoning and sense making in algebra, specifically using first-degree equations. The main purpose of the research is to investigate the forms of reasoning and sense making that pre-service mathematics teachers exhibit and use in the process of analysing problem solving, with a focus on first-degree equations. The characteristics of the tasks used for reasoning and sense making of first-degree equations, as well as the characteristics of pre-service teachers' reasoning and sense making, are identified. The study has revealed that the problem-posing task is rich in line of flight that connects intended critical aspects with reasoning and sense making, and that the pre-service teachers' use of the problem-posing task is rich in line of rupture.

This study suggests that analysing the problem posing task as a rhizomatic assemblage, with the intended critical aspects as the point of departure, gives the opportunity to identify the characteristics of reasoning and sense making. The repetition at the level of idea used in the *rhizomatic problem-posing task* (RPPT) opens abundant connections to the first-degree equation. The RPPT includes: (1) *connection* (new connections are created at every critical aspect); (2) *heterogeneity* (associating and linking elements of quite different types in relationships with critical aspect); (3) *multiplicity* (creating a whole with specific properties that cannot be explained by adding the separate properties of individual parts); (4) *rupture* (the ability of growth to start again along an old line or along a new line if an old line is broken at any place); and (5) *cartography* (no beginning or end, but all points may serve as points of departure). The RPPT can also be a task for understanding algebra instead for symbolic manipulation (e.g., Kaput, 1995; Olteanu, 2007) or forming and solving equations (e.g., Bell, 1996; Olteanu, 2015; Usiskin, 1999).

The results of the study show that the pre-service teachers create a *rhizomatic reasoning* and *sense making* that is characterized by lines of rupture. Those lines interconnect and arise from incorrect translations of RPPT into mathematical notations and the failure to discern the difference between variables and variables as unknown numbers; that is, between algebraic expression and equation. This result is in line with results from researchers concerning student misconceptions (e.g., Asquith et al., 2007; Küchemann, 1978; Powell, 2015) but, in this study, those misconceptions concern pre-service teachers instead of students. According to Işık and Kar (2012b) the main difficulties that pre-service teachers show is the inability to translate the operations and the parentheses in the equation into verbal expression. In this study, the pre- service teachers indicate difficulties with recognizing the use of operations and the parentheses in the equation as part of a rhizomatic problem-posing task.

The characteristics of reasoning in RPPT and of pre-service teachers are selecting, exploring, reconfiguring, encoding, abstracting, and connecting to highlight associations and relationships between different content. The characteristics of sense making in RPPT and of pre-service teachers are recognition, relationships, profiling, comparing, laddering, and verifying. In addition, as previously mentioned, the connection between reasoning and sense making is rich in line of flight in RPPT, while that connection is rich in line of rupture for pre-service teachers. This study unfortunately cannot uncover the effects of pre-service teachers' reasoning and sense making of RPPT on students' learning, and this is an important issue for future research.

6. Conclusion

Despite the interest in developing students' reasoning and sense making in mathematics, our knowledge remains relatively limited when it comes to making this development happen effectively. Teachers are at the heart of implementing any educational innovation or improvement of ideas in the mathematics classroom. One critical need is to investigate how pre-service teachers learn to use RPPT and how they understand students' reasoning and sense making.

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